Objectives:

- Review indeterminate forms of limits.
- Introduce and practice using l'Hospital's Rule, a derivative based tool for evaluating limits.

Recap of Indeterminate Forms

Earlier in the semester, we encountered indeterminate forms while taking limits.

For example, $\lim_{x\to\infty} \frac{x^4 - 3x^2 + 4}{x^2 + 1}$ is an indeterminate form, because:

The limit of the numerator is ∞ and the limit of the denominator is ∞ so we have the form $\frac{\infty}{\infty}$.

Note: We don't write $\lim_{x\to\infty} f(x) = \frac{\infty}{\infty}$.

Instead write $\lim_{x\to\infty} f(x)$ has indeterminate form $\frac{\infty}{\infty}$.

Types of Indeterminate Forms:

$$\frac{\infty}{\infty}, \frac{0}{0}, 0\cdot\infty, \infty - \infty, 1^{\infty}, 0^0, \infty^0$$

The terms are "fighting" over the limit. Will it be 0 or infinite or inbetween?

Forms that are <u>not</u> indeterminate:

$$\frac{0}{\infty}, \frac{\infty}{0}, \infty + \infty, -\infty - \infty, \infty^1, \frac{\text{constant}}{0}, 0^{\infty}$$

In the past we have dealt with indeterminate forms by attempting to rewrite the existing function in a way that doesn't lead to an indeterminate form.

Examples of various methods:

Find $\lim_{x\to\infty} \frac{x^4 - 3x^2 + 4}{x^2 + 1}$ if it exists. Indeterminate Form: $\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{x^4 - 3x^2 + 4}{x^2 + 1} = \lim_{x\to\infty} \frac{x^4 - 3x^2 + 4}{x^2 + 1}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{x\to\infty} \frac{x^2 - 3 + \frac{4}{x^2}}{1 + \frac{1}{x^2}} = \frac{\infty - 3 + 0}{1 + 0} = \infty$$

Find $\lim_{x\to\infty} \sqrt{x^2 + 3x} - x$ if it exists. Indeterminate form: $\infty - \infty$.

$$\lim_{x\to\infty} \sqrt{x^2+3x} - x = \lim_{x\to\infty} \sqrt{x^2+3x} - x\left(\frac{\sqrt{x^2+3x} + x}{\sqrt{x^2+3x} + x}\right) = \lim_{x\to\infty} \frac{\sqrt{x^2+3x}^2 - x^2}{\sqrt{x^2+3x} + x} = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3x} + x}$$

$$= \lim_{x\to\infty} \frac{3}{\frac{\sqrt{x^2+3x}}{x} + 1} = \lim_{x\to\infty} \frac{3}{\sqrt{\frac{x^2+3x}{x^2}} + 1} = \lim_{x\to\infty} \frac{3}{\sqrt{1 + \frac{3}{x}} + 1} = \frac{3}{\sqrt{1+0}+1} = \frac{3}{2}$$

However, there are some limits where this doesn't work. For example, $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$ has the indeterminate form $\dfrac{\infty}{\infty}$ but we can't simplify it with any of our existing strategies.

Now that we know how to take derivatives, we can add another tool to our indeterminate form toolbox:

l'Hospital's Rule: (also written l'Hôpital's Rule)

If $f(x)$ and $g(x)$ are differentiable (near a) and $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

l'Hospital's rule applies for: a is a constant, a is $\pm\infty$, two-sided or left and right limits

Why?

Think about the case where $f(a) = g(a) = 0$ (and f', g' continuous, $g'(a) \neq 0$). Near a, $f(x) \approx f'(a)(x-a) + f(a) = f'(a)(x-a)$ and $g(x) \approx g'(a)(x-a) + g(a) = g'(a)(x-a)$

so $\dfrac{f(x)}{g(x)} \approx \dfrac{f'(a)(x-a)}{g'(a)(x-a)} = \dfrac{f'(a)}{g'(a)}$.

See the book for a more proof-like explanation. The really tricky part is generalizing.

Examples:

1. $\lim\limits_{x\to\infty} \dfrac{x+3}{x^2+2x}$

   Form: $\dfrac{\infty}{\infty}$, and $\lim\limits_{x\to\infty} \dfrac{x+3}{x^2+2x} = \lim\limits_{x\to\infty} \dfrac{1}{2x+2} = 0$

2. $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4}$

   Form: $\dfrac{0}{0}$, and $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4} = \lim\limits_{x\to 2} \dfrac{1}{2x} = \frac{1}{4}$

3. $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$

   Form: $\dfrac{\infty}{\infty}$, and $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} = \lim\limits_{x\to\infty} \dfrac{2x}{e^x} = \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$

4. $\lim\limits_{x\to 0} \dfrac{x}{\cos(x)}$

   Form: $\dfrac{0}{1}$ so can't use l'Hospital. $\lim\limits_{x\to 0} \dfrac{x}{\cos(x)} = \frac{0}{1} = 0$.

5. $\lim\limits_{x\to 0} \dfrac{x}{\sin(x)}$

   Form: $\dfrac{0}{0}$, and $\lim\limits_{x\to 0} \dfrac{x}{\sin(x)} = \lim\limits_{x\to 0} \dfrac{1}{\cos(x)} = \dfrac{1}{1} = 1$